

A characterization of involutes and evolutes of a given curve in \mathbb{E}^n

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Abstract

The orthogonal trajectories of the first tangents of the curve are called the involutes of x . The hyperspheres which have higher order contact with a curve x are known osculating hyperspheres of x . The centers of osculating hyperspheres form a curve which is called generalized evolute of the given curve x in n -dimensional Euclidean space \mathbb{E}^n . In the present study, we give a characterization of involute curves of order k (resp. evolute curves) of the given curve x in n -dimensional Euclidean space \mathbb{E}^n . Further, we obtain some results on these type of curves in \mathbb{E}^3 and \mathbb{E}^4 , respectively.

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1. Introduction

Let $x = x(t) : I \subset \mathbb{R} \rightarrow \mathbb{E}^n$ be a regular curve in \mathbb{E}^n , (i.e., $\|x'(t)\| \neq 0$). Then x is called a *Frenet curve of osculating order d* , ($2 \leq d \leq n$) if $x'(t)$, $x''(t), \dots, x^{(d)}(t)$ are linearly independent and $x'(t)$, $x''(t), \dots, x^{(d+1)}(t)$ linearly dependent for all t in I [12]. In this case, $Im(x)$ lies in an d -dimensional Euclidean subspace of \mathbb{E}^{n+1} . To each Frenet curve of rank d there can be associated orthonormal d -frame $V_1 = \frac{x'(t)}{\|x'(t)\|}$, V_2, V_3, \dots, V_d along x , the Frenet

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d -frame, and $d - 1$ functions $\kappa_1, \kappa_2, \dots, \kappa_{d-1}: I \longrightarrow \mathbb{R}$, the Frenet curvature, such that

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \dots \\ V_d' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa_1 & 0 & \dots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \dots & 0 \\ 0 & -\kappa_2 & 0 & \dots & 0 \\ \dots & & & & \kappa_{d-1} \\ 0 & 0 & \dots & -\kappa_{d-1} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \dots \\ V_d \end{bmatrix} \quad (1)$$

where, $v = \|x'(t)\|$ is the speed of the curve x . In fact, to obtain $V_1, V_2, V_3, \dots, V_d$, ($2 \leq d \leq n$) it is sufficient to apply the Gram-Schmidt orthonormalization process to $x'(t), x''(t), \dots, x^{(d)}(t)$. Moreover, the functions $\kappa_1, \kappa_2, \dots, \kappa_{d-1}$ are easily obtained as by-product during this calculation.

More precisely, $V_1, V_2, V_3, \dots, V_d$ and $\kappa_1, \kappa_2, \dots, \kappa_{d-1}$ are determined by the following formulas:

$$\begin{aligned} E_1(t) &: = x'(t) \quad ; V_1 := \frac{E_1(t)}{\|E_1(t)\|}, \\ E_\alpha(t) &: = x^{(\alpha)}(t) - \sum_{i=1}^{\alpha-1} \langle x^{(\alpha)}(t), E_i(t) \rangle \frac{E_i(t)}{\|E_i(t)\|^2}, \\ V_\alpha &: = \frac{E_\alpha(t)}{\|E_\alpha(t)\|}, 2 \leq \alpha \leq n \end{aligned} \quad (2)$$

and

$$\kappa_\delta(s) := \frac{\|E_{\delta+1}(t)\|}{\|E_\delta(t)\| \|E_1(t)\|}, \quad (3)$$

respectively, where $\delta \in \{1, 2, 3, \dots, d-1\}$ (see, [2]). For the case $d = n$, the Frenet curve x is called a *generic curve* [12].

The osculating hyperplanes of a generic curve x at t is the subspace generated by $\{V_1, V_2, V_3, \dots, V_n\}$ that passes through $x(t)$. The unit vector $V_n(t)$ is called *binormal vector* of x at t . The *normal hyperplane* of x at t is defined to be the one generated by $\{V_2, V_3, \dots, V_n\}$ passing through $x(t)$ [9].

A Frenet curve of rank d for which the first Frenet curvature κ_1 is constant is called a Salkowski curve [10] (or T.C-curve [5]). Further, a Frenet curve of rank d for which $\kappa_1, \kappa_2, \dots, \kappa_{d-1}$ are constant is called (*circular*) *helix* or *W-curve* [6]. Meanwhile, a Frenet curve of rank d with constant curvature ratios $\frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_2}, \frac{\kappa_4}{\kappa_3}, \dots, \frac{\kappa_{d-1}}{\kappa_{d-2}}$ is called a *ccr-curve* (see, [8], [7]). A ccr-curve in \mathbb{E}^3 is known as generalized helix.

Given a generic curve x in \mathbb{E}^4 , the Frenet 4-frame, V_1, V_2, V_3, V_4 and the Frenet curvatures $\kappa_1, \kappa_2, \kappa_3$ are given by

$$\begin{aligned} V_1(t) &= \frac{x'(t)}{\|x'(t)\|} \\ V_4(t) &= \frac{x'(t) \wedge x''(t) \wedge x'''(t)}{\|x'(t) \wedge x''(t) \wedge x'''(t)\|} \\ V_3(t) &= \frac{V_4(t) \wedge x'(t) \wedge x''(t)}{\|V_4(t) \wedge x'(t) \wedge x''(t)\|} \\ V_2(t) &= \frac{V_3(t) \wedge V_4(t) \wedge x'(t)}{\|V_3(t) \wedge V_4(t) \wedge x'(t)\|} \end{aligned} \quad (4)$$

and

$$\kappa_1(t) = \frac{\langle V_2(t), x''(t) \rangle}{\|x'(t)\|^2}, \quad \kappa_2(t) = \frac{\langle V_3(t), x'''(t) \rangle}{\|x'(t)\|^3 \kappa_1(t)}, \quad \kappa_3(t) = \frac{\langle V_4(t), x''''(t) \rangle}{\|x'(t)\|^4 \kappa_1(t) \kappa_2(t)}. \quad (5)$$

respectively, where \wedge is the exterior product in \mathbb{E}^4 [2].

This paper is organized as follows: Section 2 gives some basic concepts of the involute curves of order k in \mathbb{E}^n . Section 3 explains some geometric properties about the involute curves of order k in \mathbb{E}^3 , where $k = 1, 2$. Section 4 tells about the involute curves of order k in \mathbb{E}^4 , where $k = 1, 2, 3$. Further these sections provides some properties and results of these type of curves. In the final section we consider generalized evolute curves in \mathbb{E}^n . Moreover, we present some results of generalized evolute curves in \mathbb{E}^3 and \mathbb{E}^4 , respectively.

2. Involute curves of order k in \mathbb{E}^n

Definition 1. Let $x = x(s)$ be a regular generic curve in \mathbb{E}^n given with the arclength parameter s (i.e., $\|x'(s)\| = 1$). Then the curves which are orthogonal to the system of k -dimensional osculating hyperplanes of x , are called the involutes of order k [1] (or, k^{th} involute [4]) of the curve x . For simplicity, we call the involutes of order 1, simply the involutes of the given curve.

In order to find the parametrization of involutes \bar{x} of order k of the curve x , we put

$$\bar{x}(s) = x(s) + \sum_{\alpha=1}^k \lambda_{\alpha}(s) V_{\alpha}(s), \quad k \leq n-1 \quad (6)$$

where λ_α is a differentiable function and s is the parameter of \bar{x} which is not necessarily an arclength parameter. The differentiation of the equation (6) and the Frenet formulae (1) give the following equation

$$\begin{aligned}\bar{x}'(s) = & (1 + \lambda'_1 - \kappa_1 \lambda_2)(s) V_1(s) \\ & + \sum_{\alpha=2}^{k-1} (\lambda'_\alpha - \lambda_{\alpha+1} \kappa_\alpha + \lambda_{\alpha-1} \kappa_{\alpha-1})(s) V_\alpha(s) \\ & + (\lambda'_k + \lambda_{k-1} \kappa_{k-1})(s) v_\alpha(s) + \kappa_k(s) \lambda_k(s) V_{k+1}(s).\end{aligned}\tag{7}$$

Furthermore, the involutes \bar{x} of order k of the curve x are determined by

$$\langle \bar{x}'(s), V_j(s) \rangle = 0, 1 \leq j \leq k \leq n-1.$$

This condition is satisfied if and only if

$$\begin{aligned}1 + \lambda'_1 - \kappa_1 \lambda_2 &= 0, \\ \lambda'_\alpha - \lambda_{\alpha+1} \kappa_\alpha + \lambda_{\alpha-1} \kappa_{\alpha-1} &= 0, \\ \lambda'_k + \lambda_{k-1} \kappa_{k-1} &= 0,\end{aligned}\tag{8}$$

where $2 \leq \alpha \leq n-1$. Consequently, the involutes of order k of a regular generic curve x are represented by the formulas (8), and when λ_α are chosen in this way, λ_k does not vanish identically and $\bar{V}_1(s) = \pm V_{k+1}$ whenever $\lambda_k \neq 0$ [4].

3. Involutives in \mathbb{E}^3

In the present section we consider involutes of order 1 and of order 2 of curves in Euclidean 3-space \mathbb{E}^3 , respectively.

3.1. Involutives of order 1 in \mathbb{E}^3

Proposition 2. *Let $x = x(s)$ be a regular curve in \mathbb{E}^3 given with nonzero Frenet curvatures κ_1 and κ_2 . Then Frenet curvatures $\bar{\kappa}_1$ and $\bar{\kappa}_2$ of the involute \bar{x} of the curve x are given by*

$$\bar{\kappa}_1 = \frac{\sqrt{\kappa_1^2 + \kappa_2^2}}{|\kappa_1| |s - c|}, \quad \bar{\kappa}_2 = \frac{\left(\frac{\kappa_2}{\kappa_1}\right)' \kappa_1^2}{(\kappa_1^2 + \kappa_2^2)(c - s)}.\tag{9}$$

Proof. Let $\bar{x} = \bar{x}(s)$ be the involute of the curve x in \mathbb{E}^3 . Then by the use of (7) with (8) we get $1 + \lambda_1'(s) = 0$, and furthermore $\lambda(s) = (c - s)$ for some integral constant c . So, we get the following parametrization

$$\bar{x}(s) = x(s) + (c - s)V_1(s). \quad (10)$$

Further, the differentiation of (10) implies that

$$\begin{aligned} \bar{x}'(s) &= \varphi V_2, \quad \varphi(s) := \lambda(s)\kappa_1(s) \\ \bar{x}''(s) &= -\varphi\kappa_1 V_1 + \varphi'V_2 + \varphi\kappa_2 V_3, \\ \bar{x}'''(s) &= -\{(\kappa_1\varphi)' + \kappa_1\varphi'\} V_1 + \{\varphi'' - \kappa_1^2\varphi - \kappa_2^2\varphi\} V_2 + \{(\kappa_2\varphi)' + \kappa_2\varphi'\} V_3. \end{aligned}$$

Now, an easy calculation gives

$$\begin{aligned} \|\bar{x}'(s)\| &= |\varphi| = |(c - s)\kappa_1|, \\ \|\bar{x}'(s) \times \bar{x}''(s)\| &= \varphi^2 \sqrt{\kappa_1^2 + \kappa_2^2}, \\ \langle \bar{x}'(s) \times \bar{x}''(s), \bar{x}'''(s) \rangle &= \varphi^3 (\kappa_1\kappa_2' - \kappa_2\kappa_1'). \end{aligned} \quad (11)$$

The parameter s is not the arc length parameter of \bar{x} , so, as is shown in [1], we have

$$\bar{\kappa}_1 = \frac{\|\bar{x}'(s) \times \bar{x}''(s)\|}{\|\bar{x}'(s)\|^3}, \quad \bar{\kappa}_2 = \frac{\langle \bar{x}'(s) \times \bar{x}''(s), \bar{x}'''(s) \rangle}{\|\bar{x}'(s) \times \bar{x}''(s)\|^2} \quad (12)$$

Hence, from the relations (11) and (12) we deduce (9). ■

By the use of (9) one can get the following result.

Corollary 3. *If $x = x(s)$ is a cylindrical helix in \mathbb{E}^3 , then the involute \bar{x} of x is a planar curve.*

3.2. Involutives of order 2 in \mathbb{E}^3

An involute of order 2 of a regular curve x in \mathbb{E}^3 has the parametrization

$$\bar{x}(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s) \quad (13)$$

where V_1, V_2 are tangent and normal vectors of x in \mathbb{E}^3 and λ_1, λ_2 are differentiable functions satisfying

$$\begin{aligned} \lambda_1'(s) &= \kappa_1(s)\lambda_2(s) - 1, \\ \lambda_2'(s) &= -\lambda_1(s)\kappa_1(s). \end{aligned} \quad (14)$$

We obtain the following result.

Proposition 4. *Let $x = x(s)$ be a regular curve in \mathbb{E}^3 given with nonzero Frenet curvatures κ_1 and κ_2 . Then Frenet curvatures $\bar{\kappa}_1$ and $\bar{\kappa}_2$ of the involute \bar{x} of order 2 of the curve x are given by*

$$\bar{\kappa}_1 = \frac{\text{sgn}(\kappa_2)}{|\lambda_2|}, \quad \bar{\kappa}_2 = \frac{\kappa_2}{\lambda_2}. \quad (15)$$

Proof. Let $\bar{x} = \bar{x}(s)$ be the involute of order 2 of the curve x in \mathbb{E}^3 . Then by the use of (7) with (8) we get

$$\bar{x}'(s) = \lambda_2(s)\kappa_2(s)V_3(s), \quad (16)$$

Further, the differentiation of (16) implies that

$$\begin{aligned} \bar{x}'(s) &= \psi(s)V_3(s), \quad \psi(s) := \lambda_2(s)\kappa_2(s) \\ \bar{x}''(s) &= -\psi(s)\kappa_2(s)V_2(s) + \psi'(s)V_3(s), \\ \bar{x}'''(s) &= -\psi(s)\kappa_1(s)\kappa_2(s)V_1(s) - \{(\psi(s)\kappa_2(s))' + \kappa_2(s)\psi'(s)\}V_2(s) \\ &\quad + \{\psi''(s) + \psi(s)\kappa_2^2(s)\}V_3(s). \end{aligned}$$

Now, an easy calculation gives

$$\begin{aligned} \|\bar{x}'(s)\| &= |\psi(s)| = |\lambda_2(s)\kappa_2(s)|, \\ \|\bar{x}'(s) \times \bar{x}''(s)\| &= \psi(s)^2\kappa_2(s), \\ \langle \bar{x}'(s) \times \bar{x}''(s), \bar{x}'''(s) \rangle &= \psi(s)^3\kappa_1(s)\kappa_2^2(s). \end{aligned} \quad (17)$$

Hence, from the relations (12) and (17) we deduce (15). ■

Corollary 5. *The involute \bar{x} of order 2 of a generalized helix in \mathbb{E}^3 is also a generalized helix in \mathbb{E}^3 .*

Solving the system of differential equations (14) we get the following result.

Corollary 6. *Let $x = x(s)$ be a unit speed Salkowski curve in \mathbb{E}^3 . Then the involute \bar{x} of order 2 of the curve x has the parametrization (13) given with the coefficient functions*

$$\begin{aligned} \lambda_1(s) &= c_1 \sin(\kappa_1 s) + c_2 \cos(\kappa_1 s), \\ \lambda_2(s) &= c_1 \cos(\kappa_1 s) - c_2 \sin(\kappa_1 s) - \frac{1}{\kappa_1}. \end{aligned} \quad (18)$$

where c_1 and c_2 are real constants.

4. Involutives in \mathbb{E}^4

In the present section we consider involutes of order k , $1 \leq k \leq 3$ of a given curve x in Euclidean 4-space \mathbb{E}^4 .

4.1. Involutives of order 1 in \mathbb{E}^4

Proposition 7. *Let $x = x(s)$ be a regular curve in \mathbb{E}^4 given with the Frenet curvatures κ_1 , κ_2 and κ_3 . Then Frenet 4-frame, $\bar{V}_1, \bar{V}_2, \bar{V}_3$ and \bar{V}_4 and Frenet curvatures $\bar{\kappa}_1$, $\bar{\kappa}_2$ and $\bar{\kappa}_3$ of the involute \bar{x} of the curve x are given by*

$$\begin{aligned}\bar{V}_1(s) &= V_2, \\ \bar{V}_2(s) &= \frac{-\kappa_1 V_1 + \kappa_2 V_3}{\sqrt{\kappa_1^2 + \kappa_2^2}}, \\ \bar{V}_3(s) &= \frac{-(\kappa_2 A - \kappa_1 C) \kappa_2 V_1 - (\kappa_2 A - \kappa_1 C) \kappa_1 V_3 + D(\kappa_1^2 + \kappa_2^2) V_4}{W \sqrt{\kappa_1^2 + \kappa_2^2}}, \\ \bar{V}_4(s) &= \frac{D \kappa_2 V_1 + D \kappa_1 V_3 - (\kappa_2 A - \kappa_1 C) V_4}{W},\end{aligned}\tag{19}$$

and

$$\begin{aligned}\bar{\kappa}_1 &= \frac{\sqrt{\kappa_1^2 + \kappa_2^2}}{|\varphi|}; \quad \varphi := (c - s)\kappa_1, \\ \bar{\kappa}_2 &= \frac{W}{\varphi^2 (\kappa_1^2 + \kappa_2^2)}, \\ \bar{\kappa}_3 &= -\frac{(\kappa_2 A - \kappa_1 C)(\kappa_3 C + D') + D(\kappa_2 A' - \kappa_1 C') + D^2 \kappa_1 \kappa_3}{W \varphi^4 \bar{\kappa}_1 \bar{\kappa}_2},\end{aligned}\tag{20}$$

respectively, where

$$\begin{aligned}A &= \kappa_1' \varphi + 2\kappa_1 \varphi' \\ C &= \kappa_2' \varphi + 2\kappa_2 \varphi' \\ D &= \kappa_2 \kappa_3 \varphi\end{aligned}$$

and

$$\begin{aligned}W &= \sqrt{D^2 (\kappa_1^2 + \kappa_2^2) + (\kappa_1 C - \kappa_2 A)^2} \\ &= |\varphi| \sqrt{\kappa_2^2 \kappa_3^2 (\kappa_1^2 + \kappa_2^2) + (\kappa_1 \kappa_2' - \kappa_2 \kappa_1')^2}.\end{aligned}\tag{21}$$

Proof. As in the proof of Proposition 2, the involute $\bar{x} = \bar{x}(s)$ of the curve x in \mathbb{E}^4 has the parametrization

$$\bar{x}(s) = x(s) + (c - s)V_1(s),$$

where V_1 is the unit tangent vector of x .

Further, the differentiation of the position vector $\bar{x}(s)$ implies that

$$\begin{aligned}\bar{x}'(s) &= \varphi V_2, \\ \bar{x}''(s) &= -\varphi\kappa_1 V_1 + \varphi' V_2 + \varphi\kappa_2 V_3, \\ \bar{x}'''(s) &= -\{(\kappa_1\varphi)' + \kappa_1\varphi'\} V_1 + \{\varphi'' - \kappa_1^2\varphi - \kappa_2^2\varphi\} V_2 \\ &\quad + \{(\kappa_2\varphi)' + \kappa_2\varphi'\} V_3 + \varphi\kappa_2\kappa_3 V_4,\end{aligned}\tag{22}$$

where $\varphi = (c - s)\kappa_1$ is a differentiable function. Consequently, substituting

$$\begin{aligned}A &= \kappa_1'\varphi + 2\kappa_1\varphi' \\ B &= \varphi'' - \kappa_1^2\varphi - \kappa_2^2\varphi \\ C &= \kappa_2'\varphi + 2\kappa_2\varphi' \\ D &= \varphi\kappa_2\kappa_3,\end{aligned}\tag{23}$$

the last vector becomes

$$\bar{x}''' = -AV_1 + BV_2 + CV_3 + DV_4.\tag{24}$$

Furthermore, differentiating \bar{x}''' with respect to s , we get

$$\begin{aligned}\bar{x}'''' &= -\{A' + \kappa_1 B\} V_1 + \{-\kappa_1 A - \kappa_2 C + B'\} V_2 \\ &\quad + \{\kappa_2 B - \kappa_3 D + C'\} V_3 + \{D' + \kappa_3 C\} V_4.\end{aligned}\tag{25}$$

Now, by the use of (22), we can compute the vector form $\bar{x}'(s) \wedge \bar{x}''(s) \wedge \bar{x}'''(s)$ and second principal normal of \bar{x} as in the following;

$$\bar{x}'(s) \wedge \bar{x}''(s) \wedge \bar{x}'''(s) = \varphi^2 \{D\kappa_2 V_1 + D\kappa_1 V_3 + (\kappa_1 C - \kappa_2 A) V_4\}$$

and

$$\bar{V}_4(s) = \frac{x'(s) \wedge x''(s) \wedge x'''(s)}{\|x'(s) \wedge x''(s) \wedge x'''(s)\|} = \frac{D\kappa_2 V_1 + D\kappa_1 V_3 - (\kappa_2 A - \kappa_1 C) V_4}{W}\tag{26}$$

where

$$W = \sqrt{D^2 (\kappa_1^2 + \kappa_2^2) + (\kappa_2 A - \kappa_1 C)^2}.\tag{27}$$

Similarly, we can compute the vector form $\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s)$ and first principal normal $\overline{V}_3(s)$ of \overline{x} as

$$\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s) = \frac{\varphi^2}{W} \left\{ -(\kappa_2 A - \kappa_1 C) \kappa_2 V_1 - (\kappa_2 A - \kappa_1 C) \kappa_1 V_3 + D (\kappa_1^2 + \kappa_2^2) V_4 \right\}$$

and

$$\begin{aligned} \overline{V}_3(s) &= \frac{\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s)}{\|\overline{V}_4(s) \wedge \overline{x}'(s) \wedge \overline{x}''(s)\|} \\ &= \frac{-(\kappa_2 A - \kappa_1 C) \kappa_2 V_1 - (\kappa_2 A - \kappa_1 C) \kappa_1 V_3 + D (\kappa_1^2 + \kappa_2^2) V_4}{W \sqrt{\kappa_1^2 + \kappa_2^2}} \end{aligned} \quad (28)$$

Finally, the vector form $\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s)$ and the normal $\overline{V}_2(s)$ of \overline{x} becomes

$$\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s) = \varphi \left\{ D^2 (\kappa_1^2 + \kappa_2^2) - (\kappa_2 A - \kappa_1 C)^2 \right\} (-\kappa_1 V_1 + \kappa_2 V_3)$$

and

$$\overline{V}_2(s) = \frac{\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s)}{\|\overline{V}_3(s) \wedge \overline{V}_4(s) \wedge \overline{x}'(s)\|} = \frac{-\kappa_1 V_1 + \kappa_2 V_3}{\sqrt{\kappa_1^2 + \kappa_2^2}}. \quad (29)$$

Consequently, an easy calculation gives

$$\begin{aligned} \langle \overline{V}_2(s), \overline{x}''(s) \rangle &= \varphi \sqrt{\kappa_1^2 + \kappa_2^2} \\ \langle \overline{V}_3(s), \overline{x}'''(s) \rangle &= \frac{W}{\sqrt{\kappa_1^2 + \kappa_2^2}} \\ \langle \overline{V}_4(s), \overline{x}''''(s) \rangle &= -\frac{(\kappa_2 A - \kappa_1 C) (\kappa_3 C + D') + D (\kappa_2 A' - \kappa_1 C') + D^2 \kappa_1 \kappa_3}{W}. \end{aligned} \quad (30)$$

Hence, from the relations (30) and (5) we deduce (20). This completes the proof of the proposition. ■

For the case x is a W -curve one can get the following results.

Corollary 8. [11] *Let \overline{x} be an involute of a generic x curve in \mathbb{E}^4 given with the Frenet curvatures $\overline{\kappa}_1, \overline{\kappa}_2$ and $\overline{\kappa}_3$. If x is a W -curve then the Frenet 4-frame, $\overline{V}_1, \overline{V}_2, \overline{V}_3$ and \overline{V}_4 and the Frenet curvatures $\overline{\kappa}_1, \overline{\kappa}_2$ and $\overline{\kappa}_3$ of the*

involute \bar{x} of the curve x are given by

$$\begin{aligned}\bar{V}_1(s) &= V_2, \\ \bar{V}_2(s) &= \frac{-\kappa_1 V_1 + \kappa_2 V_3}{\sqrt{\kappa_1^2 + \kappa_2^2}} \\ \bar{V}_3(s) &= V_4 \\ \bar{V}_4(s) &= \frac{\kappa_2 V_1 + \kappa_1 V_3}{\sqrt{\kappa_1^2 + \kappa_2^2}},\end{aligned}\tag{31}$$

and

$$\begin{aligned}\bar{\kappa}_1 &= \frac{\sqrt{\kappa_1^2 + \kappa_2^2}}{|\varphi|}, \\ \bar{\kappa}_2 &= \frac{\kappa_2 \kappa_3}{|\varphi| \sqrt{\kappa_1^2 + \kappa_2^2}}, \\ \bar{\kappa}_3 &= \frac{-\kappa_1 \kappa_3}{|\varphi| \sqrt{\kappa_1^2 + \kappa_2^2}}\end{aligned}\tag{32}$$

respectively, where $\varphi = (c - s)\kappa_1$.

Corollary 9. *Let \bar{x} be an involute of a generic x curve in \mathbb{E}^4 given with the Frenet curvatures $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$. If x is a W -curve then \bar{x} becomes a ccr -curve.*

4.2. Involutes of order 2 in \mathbb{E}^4

An involute of order 2 of a regular curve x in \mathbb{E}^4 has the parametrization

$$\bar{x}(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s)\tag{33}$$

where V_1, V_2 are tangent and normal vectors of x in \mathbb{E}^4 and λ_1, λ_2 are differentiable functions satisfying

$$\begin{aligned}\lambda_1'(s) &= \kappa_1(s)\lambda_2(s) - 1, \\ \lambda_2'(s) &= -\lambda_1(s)\kappa_1(s).\end{aligned}\tag{34}$$

As in the previous subsection we get the following result.

Corollary 10. *Let $x = x(s)$ be a unit speed Salkowski curve in \mathbb{E}^4 . Then the involute \bar{x} of order 2 of the curve x has the parametrization (33) given with the coefficient functions*

$$\begin{aligned}\lambda_1(s) &= c_1 \sin(\kappa_1 s) + c_2 \cos(\kappa_1 s), \\ \lambda_2(s) &= c_1 \cos(\kappa_1 s) - c_2 \sin(\kappa_1 s) - \frac{1}{\kappa_1}.\end{aligned}\tag{35}$$

where c_1 and c_2 are real constants.

We obtain the following result.

Proposition 11. *Let $x = x(s)$ be a regular curve in \mathbb{E}^4 given with nonzero Frenet curvatures κ_1, κ_2 and κ_3 . Then Frenet 4-frame, $\bar{V}_1, \bar{V}_2, \bar{V}_3$ and \bar{V}_4 and Frenet curvatures $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$ of the involute \bar{x} of order 2 of a regular curve x in \mathbb{E}^4 are given by*

$$\begin{aligned}\bar{V}_1(s) &= V_3, \\ \bar{V}_2(s) &= \frac{-\kappa_2 V_2 + \kappa_3 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}}, \\ \bar{V}_3(s) &= \frac{K(\kappa_2^2 + \kappa_3^2)V_1 + (\kappa_2 N - \kappa_3 L)\kappa_3 V_2 + (\kappa_2 N - \kappa_3 L)\kappa_2 V_4}{W\sqrt{\kappa_2^2 + \kappa_3^2}}, \\ \bar{V}_4(s) &= \frac{(\kappa_2 N - \kappa_3 L)V_1 + \kappa_3 K V_2 + \kappa_2 K V_4}{W},\end{aligned}\tag{36}$$

and

$$\begin{aligned}\bar{\kappa}_1 &= \frac{\sqrt{\kappa_2^2 + \kappa_3^2}}{|\phi|}; \quad \phi := \lambda_2(s)\kappa_2(s) \\ \bar{\kappa}_2 &= \frac{W}{\phi^2(\kappa_2^2 + \kappa_3^2)}, \\ \bar{\kappa}_3 &= \frac{(\kappa_2 N - \kappa_3 L)(\kappa_1 L + K') + (\kappa_2 N' - \kappa_3 L')K + \kappa_1 \kappa_3 K^2}{W\phi^4 \bar{\kappa}_1 \bar{\kappa}_2}\end{aligned}\tag{37}$$

where

$$\begin{aligned}K &= \kappa_1 \kappa_2 \phi \\ L &= 2\kappa_2 \phi' + \kappa_2' \phi \\ N &= 2\kappa_3 \phi' + \kappa_3' \phi\end{aligned}$$

and

$$\begin{aligned} W &= \sqrt{K^2 (\kappa_2^2 + \kappa_3^2) + (\kappa_2 N - \kappa_3 L)^2} \\ &= |\phi| \sqrt{\kappa_1^2 \kappa_2^2 (\kappa_2^2 + \kappa_3^2) + (\kappa_2 \kappa_3' - \kappa_3 \kappa_2')^2}. \end{aligned} \quad (38)$$

Proof. Let $\bar{x} = \bar{x}(s)$ be the involute of order 2 of the curve x in \mathbb{E}^4 . Then by the use of (7), we get

$$\bar{x}'(s) = \phi V_3 \quad (39)$$

where $\phi = \lambda_2(s)\kappa_2(s)$ is a differentiable function. Further, the differentiation of (39) implies that

$$\begin{aligned} \bar{x}''(s) &= -\phi \kappa_2 V_2 + \phi' V_3 + \phi \kappa_3 V_4, \\ \bar{x}'''(s) &= \kappa_1 \kappa_2 \phi V_1 + \{2\kappa_2 \phi' + \kappa_2' \phi\} V_2, \\ &\quad + \{\phi'' - \kappa_2^2 \phi - \kappa_3^2 \phi\} V_3 + \{2\kappa_3 \phi' + \kappa_3' \phi\} V_4. \end{aligned} \quad (40)$$

Consequently, substituting

$$\begin{aligned} K &= \kappa_1 \kappa_2 \phi \\ L &= 2\kappa_2 \phi' + \kappa_2' \phi \\ M &= \phi'' - \kappa_2^2 \phi - \kappa_3^2 \phi \\ N &= 2\kappa_3 \phi' + \kappa_3' \phi \end{aligned} \quad (41)$$

the last vector becomes

$$\bar{x}''' = KV_1 - LV_2 + MV_3 + NV_4. \quad (42)$$

Furthermore, differentiating \bar{x}''' with respect to s we get

$$\begin{aligned} \bar{x}'''' &= \{K' + \kappa_1 L\} V_1 + \{\kappa_1 K - \kappa_2 M - L'\} V_2 \\ &\quad + \{M' - \kappa_2 L - \kappa_3 N\} V_3 + \{N' + \kappa_3 M\} V_4 \end{aligned} \quad (43)$$

Hence, substituting (39)-(43) into (4) and (5), after some calculations as in the previous proposition, we get the result. ■

For the case x is a W -curve then one can get the following results.

Corollary 12. *Let \bar{x} be an involute of order 2 of a generic x curve in \mathbb{E}^4 given with the Frenet curvatures $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$. If x is a W -curve then the*

Frenet 4-frame, $\bar{V}_1, \bar{V}_2, \bar{V}_3$ and \bar{V}_4 and Frenet curvatures $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$ of the involute \bar{x} of order 2 of a regular curve x in \mathbb{E}^4 are given by

$$\begin{aligned}\bar{V}_1(s) &= V_3, \\ \bar{V}_2(s) &= \frac{-\kappa_2 V_2 + \kappa_3 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}} \\ \bar{V}_3(s) &= V_1 \\ \bar{V}_4(s) &= \frac{\kappa_3 V_2 + \kappa_2 V_4}{\sqrt{\kappa_2^2 + \kappa_3^2}},\end{aligned}\tag{44}$$

and

$$\begin{aligned}\bar{\kappa}_1 &= \frac{\sqrt{\kappa_2^2 + \kappa_3^2}}{|\phi|}, \\ \bar{\kappa}_2 &= \frac{\kappa_1 \kappa_2}{|\phi| \sqrt{\kappa_2^2 + \kappa_3^2}}, \\ \bar{\kappa}_3 &= \frac{\kappa_1 \kappa_3}{|\phi| \sqrt{\kappa_2^2 + \kappa_3^2}},\end{aligned}\tag{45}$$

holds, where $\phi(s) = \lambda_2(s)\kappa_2(s)$.

Corollary 13. *Let \bar{x} be an involute of order 2 of a generic x curve in \mathbb{E}^4 given with the Frenet curvatures $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$. If x is a W -curve then \bar{x} becomes a ccr -curve.*

4.3. Involutives of order 3 in \mathbb{E}^4

An involute of order 3 of a regular curve x in \mathbb{E}^4 has the parametrization

$$\bar{x}(s) = x(s) + \lambda_1(s)V_1(s) + \lambda_2(s)V_2(s) + \lambda_3(s)V_3(s)\tag{46}$$

where

$$\begin{aligned}\lambda_1'(s) &= \kappa_1(s)\lambda_2(s) - 1, \\ \lambda_2'(s) &= \lambda_3\kappa_2 - \lambda_1\kappa_1 \\ \lambda_3'(s) &= -\lambda_2(s)\kappa_2(s).\end{aligned}\tag{47}$$

By solving the system of differential equations in (47) we get the following result.

Corollary 14. *Let $x = x(s)$ is a unit speed W -curve in \mathbb{E}^4 . Then the involute \bar{x} of order 3 of the curve x has the parametrization (46) given with the coefficient functions*

$$\begin{aligned}\lambda_1(s) &= \frac{\kappa_1 (c_2 \sin(\kappa s) - c_3 \cos(\kappa s))}{\sqrt{\kappa}} + \frac{c_1 \kappa - \kappa_2^2 s}{\kappa}, \\ \lambda_2(s) &= c_2 \cos(\kappa s) - c_3 \sin(\kappa s) + \frac{\kappa_1}{\kappa}, \\ \lambda_3(s) &= \frac{\kappa_2 (c_2 \sin(\kappa s) - c_3 \cos(\kappa s))}{\sqrt{\kappa}} - \frac{c_1 \kappa_1 \kappa - \kappa_1 \kappa_2^2 s}{\kappa \kappa_2},\end{aligned}\tag{48}$$

where $\kappa = \kappa_1^2 + \kappa_2^2$, c_1 , c_2 and c_3 are real constants.

We obtain the following result.

Proposition 15. *Let $x = x(s)$ be a regular curve in \mathbb{E}^4 given with nonzero Frenet curvatures κ_1, κ_2 and κ_3 . Then Frenet Frenet 4-frame, $\bar{V}_1, \bar{V}_2, \bar{V}_3$ and \bar{V}_4 and Frenet curvatures $\bar{\kappa}_1, \bar{\kappa}_2$ and $\bar{\kappa}_3$ of the involute \bar{x} of order 3 of a regular curve x in \mathbb{E}^4 are given by*

$$\begin{aligned}\bar{V}_1(s) &= V_4, \\ \bar{V}_2(s) &= -V_3, \\ \bar{V}_3(s) &= V_2, \\ \bar{V}_4(s) &= V_1,\end{aligned}\tag{49}$$

and

$$\begin{aligned}\bar{\kappa}_1 &= \frac{\kappa_3}{|\psi|}, \\ \bar{\kappa}_2 &= \frac{\kappa_2}{|\psi|}, \\ \bar{\kappa}_3 &= -\frac{\kappa_1}{|\psi|},\end{aligned}\tag{50}$$

where $\psi(s) = \lambda_3(s)\kappa_3(s)$.

Proof. Let $\bar{x} = \bar{x}(s)$ be the involute of order 3 of the curve x in \mathbb{E}^4 . Then by the use of (7) with (8), we get

$$\bar{x}'(s) = \psi V_4\tag{51}$$

where $\psi = \lambda_3(s)\kappa_3(s)$ is a differentiable function. Further, the differentiation of (51) implies that

$$\begin{aligned}\bar{x}''(s) &= -\psi\kappa_3V_3 + \psi'V_4, \\ \bar{x}'''(s) &= \kappa_2\kappa_3\psi V_2 - \{2\kappa_3'\psi + \kappa_3'\phi\}V_3 + \{\psi'' - \kappa_3^2\psi\}V_4.\end{aligned}$$

Consequently, substituting

$$\begin{aligned}E &= \kappa_2\kappa_3\psi \\ F &= 2\kappa_3'\psi + \kappa_3'\phi \\ G &= \psi'' - \kappa_3^2\psi\end{aligned}\tag{52}$$

the last vector becomes

$$\bar{x}''' = EV_2 - FV_3 + GV_4.\tag{53}$$

Furthermore, differentiating \bar{x}''' with respect to s we get

$$\begin{aligned}\bar{x}'''' &= -\kappa_1EV_1 + \{\kappa_2F + E'\}V_2 \\ &\quad + \{\kappa_2E - \kappa_3G - F'\}V_3 + \{G' - \kappa_3F\}V_4.\end{aligned}\tag{54}$$

Hence, substituting (51)-(54) into (4) and (5), after some calculations we get the result. ■

Corollary 16. *The involute \bar{x} of order 3 of a ccr-curve x in \mathbb{E}^4 is also a ccr-curve of \mathbb{E}^4 .*

5. Generalized Evolute Curves in \mathbb{E}^{m+1}

Let $x = x(s)$ be a generic curve in \mathbb{E}^n given with Frenet frame $V_1, V_2, V_3, \dots, V_n$ and Frenet curvatures $\kappa_1, \kappa_2, \dots, \kappa_{n-1}$. For simplicity, we can take $n = m + 1$, to construct the Frenet frame $V_1 = T, V_2 = N_1, V_3 = N_2, \dots, V_n = N_m$ and Frenet curvatures $\kappa_1, \kappa_2, \dots, \kappa_m$. The centre of the osculating hypersphere of x at a point lies in the hyperplane normal to the x at that point. The curve passing through the centers of the osculating hyperspheres of x defined by

$$\tilde{x} = x + \sum_{i=1}^m c_i N_i,\tag{55}$$

which is called *generalized evolute* (or focal curve) of x , where c_1, c_2, \dots, c_m are smooth functions of the parameter of the curve x . The function c_i is called the i^{th} focal curvature of γ . Moreover, the function c_1 never vanishes and $c_1 = \frac{1}{k_1}$ [13].

The differentiation of the equation (55) and the Frenet formulae (1) give the following equation

$$\begin{aligned} \tilde{x}'(s) = & (1 - \kappa_1 c_1)T + (c'_1 - \kappa_2 c_2)N_1 + \\ & + \sum_{i=2}^{m-1} (c_{i-1}\kappa_i + c'_i - c_{i+1}\kappa_{i+1})N_i + (c_{m-1}\kappa_m + c'_m)N_m. \end{aligned} \quad (56)$$

Since, the osculating planes of \tilde{x} are the normal planes of x , and the points of \tilde{x} are the center of the osculating sphere of x then the generalized evolutes \tilde{x} of the curve x are determined by

$$\langle \tilde{x}'(s), T(s) \rangle = \langle \tilde{x}'(s), N_1(s) \rangle = \dots = \langle \tilde{x}'(s), N_{m-1}(s) \rangle = 0. \quad (57)$$

This condition is satisfied if and only if

$$\begin{aligned} 1 - \kappa_1 c_1 &= 0 \\ c'_1 - \kappa_2 c_2 &= 0 \\ &\vdots \\ c_{i-1}\kappa_i + c'_i - c_{i+1}\kappa_{i+1} &= 0, \quad 2 \leq i \leq m-1. \end{aligned} \quad (58)$$

hold. So, the focal curvatures of a curve parametrized by arclength s satisfy the following "scalar Frenet equation" for $c_m \neq 0$:

$$\frac{R_m^2}{2c_m} = c_{m-1}\kappa_m + c'_m \quad (59)$$

where

$$R_m = \|\tilde{x} - x\| = \sqrt{c_1^2 + c_2^2 + \dots + c_m^2}$$

is the radius of the osculating m -sphere [13]. Consequently, the generalized evolutes \tilde{x} of the curve x are represented by the formulas (55), and

$$\tilde{x}'(s) = (c_{m-1}\kappa_m + c'_m)N_m. \quad (60)$$

If $\tilde{x}'(s) = 0$, then R_m is constant and the curve x is spherical.

Proposition 17. [13] *The curvatures of a generic curve $x = x(s) : I \subset \mathbb{R} \rightarrow \mathbb{E}^{m+1}$ parametrized by arc length, may be obtained in terms of the focal curvatures by the formula:*

$$\kappa_i = \frac{c_1 c'_1 + c_2 c'_2 + \dots + c_{i-1} c'_{i-1}}{c_{i-1} c_i}. \quad (61)$$

Remark 18. *For a generic curve, the functions c_i or c_{i-1} can vanish at isolated points. At these points the function $c_1 c'_1 + c_2 c'_2 + \dots + c_{i-1} c'_{i-1}$ also vanishes, and the corresponding value of the function κ_i may be obtained by l'Hospital rule. Denote by R_m the radius of the osculating m -sphere. Obviously $R_m^2 = c_1^2 + c_2^2 + \dots + c_m^2$ [13].*

Theorem 19. [13] *Let $x = x(s)$ be a generic curve in \mathbb{E}^{m+1} given with Frenet frame T, N_1, N_2, \dots, N_m and Frenet curvatures $\kappa_1, \kappa_2, \dots, \kappa_m$. Then Frenet frame $\tilde{T}, \tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_m$ and Frenet curvatures $\tilde{\kappa}_1, \tilde{\kappa}_2, \dots, \tilde{\kappa}_m$ of the generalized evolute \tilde{x} of x in \mathbb{E}^{m+1} are given by*

$$\begin{aligned} \tilde{T} &= \epsilon N_m \\ \tilde{N}_k &= \delta_k N_{m-k}; \quad 1 \leq k \leq m-1 \\ \tilde{N}_m &= \pm T \end{aligned} \quad (62)$$

and

$$\frac{\tilde{\kappa}_1}{|\kappa_m|} = \frac{\tilde{\kappa}_2}{\kappa_{m-1}} = \dots = \frac{|\tilde{\kappa}_m|}{\kappa_1} = \frac{1}{|c_{m-1} \kappa_m + c'_m|} \quad (63)$$

where $\epsilon(s)$ is the sign of $(c_{m-1} \kappa_m + c'_m)(s)$ and δ_k the sign of $(-1)^k \epsilon(s) \kappa_m(s)$.

5.1. Evolutes in \mathbb{E}^3

An generalized evolute of a regular curve x in \mathbb{E}^3 has the parametrization

$$\tilde{x}(s) = x(s) + c_1(s) N_1(s) + c_2(s) N_2(s) \quad (64)$$

where N_1 and N_2 are normal vectors of x in \mathbb{E}^3 and c_1, c_2 are focal curvatures satisfying

$$c_1(s) = \frac{1}{\kappa_1(s)}, \quad c_2(s) = \frac{\rho'(s)}{\kappa_2(s)}. \quad (65)$$

where $\rho = c_1 = \frac{1}{\kappa_1}$ is the radius of the curvature of x .

We obtain the following result.

Proposition 20. *Let $x = x(s)$ be a regular curve in \mathbb{E}^3 given with nonzero Frenet curvatures κ_1 and κ_2 . Then Frenet curvatures $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ of the evolute \tilde{x} of the curve x are given by*

$$\tilde{\kappa}_1 = \frac{\kappa_2^2}{|\rho\kappa_2^2 + \rho'|}, \quad \tilde{\kappa}_2 = \frac{\kappa_1\kappa_2}{|\rho\kappa_2^2 + \rho'|}. \quad (66)$$

where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of x .

Proof. As a consequence of (63) we get (64). ■

Corollary 21. *The evolute \tilde{x} of a generalized helix in \mathbb{E}^3 is also a generalized helix in \mathbb{E}^3 .*

By the use of (59) with (65) one can get the following result.

Corollary 22. *A regular curve with nonzero curvatures κ_1 and κ_2 lies in a sphere if and only if*

$$\left(\frac{\rho'}{\kappa_2}\right)' + \rho\kappa_2 = 0 \quad (67)$$

holds, where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of x .

5.2. Evolutes in \mathbb{E}^4

An generalized evolute of a generic curve x in \mathbb{E}^4 has the parametrization

$$\tilde{x}(s) = x(s) + c_1(s)N_1(s) + c_2(s)N_2(s) + c_3(s)N_3(s) \quad (68)$$

where N_1 , N_2 and N_3 are normal vectors of x in \mathbb{E}^4 and c_1 , c_2 and c_3 are focal curvatures satisfying

$$c_1(s) = \frac{1}{\kappa_1(s)}, \quad c_2(s) = \frac{\rho'(s)}{\kappa_2(s)}, \quad c_3(s) = \frac{\rho(s)\kappa_2(s) + \left(\frac{\rho'(s)}{\kappa_2(s)}\right)'}{\kappa_3(s)}. \quad (69)$$

where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature of x .

We obtain the following result.

Proposition 23. *Let $x = x(s)$ be a regular curve in \mathbb{E}^4 given with nonzero Frenet curvatures κ_1, κ_2 and κ_3 . Then Frenet 4-frame, $\tilde{T}, \tilde{N}_1, \tilde{N}_2$ and \tilde{N}_3 and Frenet curvatures $\tilde{\kappa}_1, \tilde{\kappa}_2$ and $\tilde{\kappa}_3$ of the evolute \tilde{x} of a regular curve x in \mathbb{E}^4 are given by*

$$\begin{aligned}\tilde{T}(s) &= N_3, \\ \tilde{N}_1(s) &= -N_2, \\ \tilde{N}_2(s) &= N_1, \\ \tilde{N}_3(s) &= T,\end{aligned}\tag{70}$$

and

$$\begin{aligned}\tilde{\kappa}_1 &= \frac{\kappa_3}{|\psi|}, \\ \tilde{\kappa}_2 &= \frac{\kappa_2}{|\psi|}, \\ \tilde{\kappa}_3 &= -\frac{\kappa_1}{|\psi|}\end{aligned}\tag{71}$$

where $\psi(s) = c_2(s)\kappa_3(s) + c'_3(s)$ is a smooth function.

Proof. As a consequence of (62) with (63) we get the result. ■

Corollary 24. *The evolute \tilde{x} of a ccr-curve x in \mathbb{E}^4 is also a ccr-curve of \mathbb{E}^4 .*

By the use of (60) with (65) one can get the following result.

Corollary 25. *A regular curve with nonzero curvatures κ_1, κ_2 and κ_3 lies on a sphere if and only if*

$$\left(\frac{\rho(s)\kappa_2(s) + \left(\frac{\rho'(s)}{\kappa_2(s)} \right)'}{\kappa_3(s)} \right)' + \rho'(s) \frac{\kappa_3(s)}{\kappa_2(s)} = 0\tag{72}$$

holds, where $\rho = \frac{1}{\kappa_1}$ is the radius of the curvature.

Proposition 26. [7] *A curve $x = x(s) : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ is spherical, i.e., it is contained in a sphere of radius R , if and only if x can be decomposed as*

$$x(s) = m - \frac{R}{\kappa_1} N_1(s) + \frac{R\kappa'_1}{\kappa_2\kappa_1^2} N_2(s) + \frac{R}{\kappa_3} \left(\frac{\kappa'_1}{\kappa_2\kappa_1^2} \right)' N_3(s).\tag{73}$$

where m is the center of the sphere.

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